

Raman Scattering by a Two-Dimensional Fermi Liquid with Spin-Orbit Coupling

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(Dated: January 5, 2017)

We present a microscopic theory of Raman scattering by a two-dimensional Fermi liquid (FL) with Rashba and Dresselhaus types of spin-orbit coupling, and subject to an in-plane magnetic field (\vec{B}). In the long-wavelength limit, the Raman spectrum probes the collective modes of such a FL: the chiral spin waves. The characteristic features of these modes are a linear-in- q term in the dispersion and the dependence of the mode frequency on the directions of both \vec{q} and \vec{B} . All of these features have been observed in recent Raman experiments on CdTe quantum wells.

Raman scattering of light is a unique tool that allows to study dynamics of elementary excitations in solids both in space and time and to probe both single-particle and collective properties of electron systems. It helps to understand a variety of phenomena: from pairing mechanisms in high-temperature superconductors[1] to spin waves for in spintronic devices[2]. The latter requires probing dynamics of electron spins, which can be done if the incident and scattered light are polarized perpendicular to each other (cross-polarized geometry)[3].

Non-trivial spin dynamics is encountered in systems with spontaneous magnetic order, or in an external magnetic field, or else in the presence of spin-orbit coupling (SOC). In this Letter, we develop a microscopic theory of Raman scattering by a two-dimensional (2D) Fermi liquid (FL) in the presence of an in-plane magnetic field and both Rashba [4] and Dresselhaus [5] types of SOC. Breaking spin conservation leads to a substantial modification of the Raman spectrum already at the non-interacting level [6]. Unlike in the $SU(2)$ -invariant case, the cross-section of Raman scattering cannot be expressed only in terms of charge and spin susceptibilities. We show, however, that the scattering cross-section for the cross-polarized geometry in the long-wavelength limit is parameterized by components of the spin susceptibility tensor. In a FL, these components contain poles that correspond to collective modes arising from an interplay between electron-electron interaction (eei) and dynamics of spins.

Recently, a dispersing peak was observed in resonant Raman scattering on magnetically doped CdTe quantum wells in the presence of an in-plane magnetic field [7–9]. The telling signs of the interplay between eei and SOC are the linear-in- q term in the dispersion and the π -periodic modulation of the spectrum as the magnetic field is rotated in the plane of 2D electron gas (2DEG). We identify the observed peak with one of the long sought after chiral spin waves (CSWs) [10–13]. These waves are collective oscillations of the magnetization (\vec{M}) that exist even in the absence of the magnetic field. In zero field, there are three such modes [11–13] which are massive, i.e., their frequencies are finite at $q \rightarrow 0$, and disperse with q on a characteristic scale set by spin-orbit splitting. The

modes are linearly polarized with \vec{M} being in the 2DEG plane for two of the modes and along the normal for the third one. If the in-plane magnetic field (\vec{B}) is applied, the mode with $\vec{M} \parallel \vec{B}$ remains linearly polarized while the other two modes with $\vec{M} \perp \vec{B}$ become elliptically polarized [14]. Figure 1a depicts the evolution of the excitation spectrum with B at $q = 0$. As the field increases, two out of the three modes run into the continuum of spin-flip excitations (SFE), while the third one merges with the continuum at $B = B_c$, when the spin-split Fermi surfaces (FSs) become degenerate, and re-emerges to the right of this point. As the field is increased further, this mode transforms gradually into the Silin-Leggett (SL) mode of a partially polarized FL [15–17]. In the experiment, the effective Zeeman energy is larger than both Rashba and Dresselhaus splittings which, according to Fig. 1a, allows us to focus on the case of a single CSW adiabatically connected to the SL mode. We show that, at small q , the dispersion of this mode can be written as

$$\Omega(\vec{q}, \vec{B}) = \Omega(\vec{0}, \theta_{\vec{B}}) + w(\theta_{\vec{q}}, \theta_{\vec{B}})q + Aq^2, \quad (1)$$

where $\theta_{\vec{x}}$ is the azimuthal angle of \vec{x} . Symmetry dictates that Rashba and Dresselhaus SOC contribute $\sin(\theta_{\vec{q}} - \theta_{\vec{B}})$ and $\cos(\theta_{\vec{q}} + \theta_{\vec{B}})$ terms to $w(\theta_{\vec{q}}, \theta_{\vec{B}})$, correspondingly, while the mass term, $\Omega(\vec{0}, \theta_{\vec{B}})$, and the boundaries of the SFE continuum are π -periodic function of $\theta_{\vec{B}}$.

In general, Raman scattering probes both charge and spin excitations. However, if the polarizations of the incident (\vec{e}_i) and scattered (\vec{e}_s) light are perpendicular to each other, the Raman vertex couples directly to the electron spin [3, 6]. The Feynman diagrams for the differential scattering cross-section are shown in Fig. 2. The first diagram on the left-hand side is the pure spin part, while the spin-charge part arising due to SOC is represented by a sum of bubbles connected by the Coulomb interaction (dashed line). In the long wavelength limit ($\vec{q} \rightarrow 0$), the spin-charge part vanishes by charge conservation; at small but finite \vec{q} , the spin part is still larger than the spin-charge one. For non-interacting electrons the contribution of the former to the scattering cross-section can

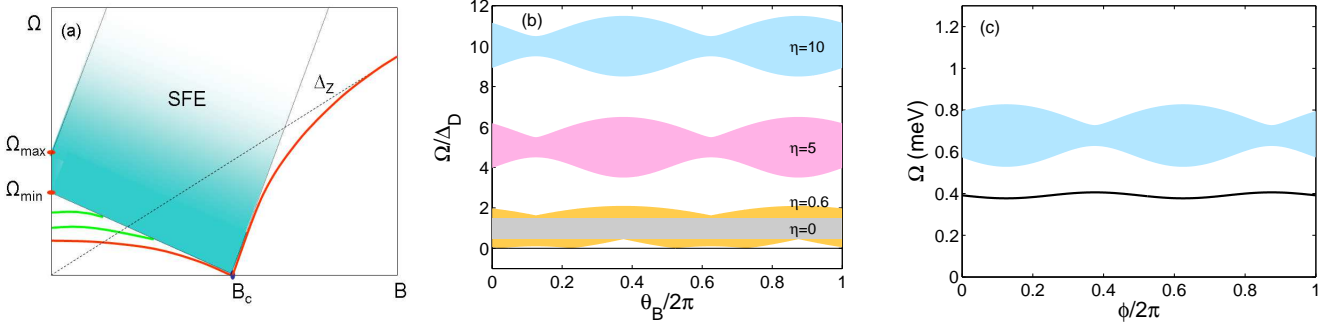


FIG. 1: Color online: a) Schematically: excitation spectrum (at $q = 0$) of a 2D Fermi liquid with spin-orbit coupling and subject to in-plane magnetic field \vec{B} . Shaded region: continuum of single-particle spin-flip excitations; lines: chiral spin waves. b) Continuum as a function of the angle $\theta_{\vec{B}}$ between \vec{B} and the (100) direction. $\Delta_R/\Delta_D = 0.5$, $\eta = \Delta_Z/\Delta_D$, where $\Delta_{D/R/Z}$ is Dresselhaus/Rashba/Zeeman splitting. c) Continuum (shaded) and collective-mode frequency at $q = 0$ (line) as a function of $\phi = \theta_{\vec{B}} - \pi/2$ for $\text{Cd}_{1-x}\text{Mn}_x\text{Te}$ quantum well.

be written as [6]

$$\frac{d^2 \mathcal{A}}{d\Omega d\mathcal{O}} \propto \sum_{\mu\mu'} |\gamma_{\mu\mu'}|^2 \text{Im}[L_{\mu\mu'}^0]; \quad \gamma_{\mu\mu'} = \langle \vec{n} \cdot \hat{\sigma} e^{i\vec{q} \cdot \vec{r}} \rangle_{\mu\mu'}, \quad (2)$$

where $d\mathcal{O}$ is the solid-angle element, $\hat{\sigma}$ are the Pauli matrices, $L_{\mu\mu'}^0 \equiv [f(\varepsilon_\mu) - f(\varepsilon_{\mu'})]/(\Omega + \varepsilon_\mu - \varepsilon_{\mu'} + i0^+)$, ε_μ is the energy of state μ , $f(\varepsilon)$ is the Fermi function, and $\vec{n} = \vec{e}_1 \times \vec{e}_3$ [18]. At small \vec{q} , the off-diagonal (spin-flip) processes dominate the cross-section. In the most general case, when both time-reversal and inversion symmetries are broken, the cross-section is given by [19]

$$\frac{d^2 \mathcal{A}}{d\Omega d\mathcal{O}} \propto \text{Im} \sum_{i,j \in \{1,2\}} n_i n_j \chi_{ij}^0(\vec{q}, \Omega) + n_3^2 \chi_{33}^0(\vec{q}, \Omega), \quad (3)$$

where $\chi_{lm}^0(\vec{q}, \Omega)$ is the spin-spin correlation function of non-interacting electrons (the x_1 and x_2 axes are in the 2DEG plane and the x_3 axis is along the normal). In the absence of SOC, the cross-section contains only the transverse (to the direction of \vec{B}) spin-spin correlation function, $\chi_{\perp}^0 = \chi_{+-}^0 + \chi_{-+}^0$ [20, 21]. In the presence of SOC, however, the result does not reduce to χ_{\perp}^0 . Instead, the partial components of χ_{lm}^0 contribute to the cross-section with weights determined by the direction of \vec{n} .

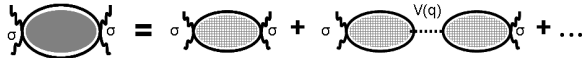


FIG. 2: Diagrams for the differential cross-section of Raman scattering in cross-polarized geometry. Shaded bubbles denote vertex corrections due to eei . $V(q)$ is the coulomb interaction.

In the presence of eei , the free-electron correlation function in Eq. (3) is replaced by the renormalized one: $\chi_{lm}^0 \rightarrow \chi_{lm}$. The poles of χ_{lm} correspond to collective

modes. In general, equations of motion for the three components of magnetization are coupled to each other, and hence all the components of $\hat{\chi}$ have the same poles but with different residues.

To proceed further, we assume that the magnitudes of the Rashba (Δ_R), Dresselhaus (Δ_D), and Zeeman (Δ_Z) energy splittings are much smaller than the Fermi energy [22]. The non-interacting Hamiltonian for a (001) quantum well can then be written as

$$\mathcal{H}_{\vec{k}} = v_F(k - k_F)\hat{\sigma}_0 + \sum_{i=1,2} s_i \hat{\sigma}_i, \quad (4)$$

where $\hat{\sigma}_0$ is the identity matrix, the x_1 axis is chosen along the (100) direction, v_F and k_F are the Fermi velocity and momentum in the absence of SOC and magnetic field, correspondingly, and

$$\begin{aligned} s_1 &= \frac{1}{2} (\Delta_R \sin \theta_{\vec{k}} + \Delta_D \cos \theta_{\vec{k}} + \Delta_Z \cos \theta_{\vec{B}}), \\ s_2 &= -\frac{1}{2} (\Delta_R \cos \theta_{\vec{k}} + \Delta_D \sin \theta_{\vec{k}} - \Delta_Z \sin \theta_{\vec{B}}). \end{aligned} \quad (5)$$

The corresponding Green's function is given by

$$\hat{G}_K^0 = \frac{g_+ + g_-}{2} \hat{\sigma}_0 + \frac{g_+ - g_-}{2|s|} \sum_{i=1,2} s_i \hat{\sigma}_i, \quad (6a)$$

$$g_{\pm} = \frac{1}{i\omega_n - \varepsilon_{\vec{k}, \pm}}, \quad \varepsilon_{\vec{k}, \pm} = v_F(k - k_F) \pm |s|, \quad (6b)$$

where $|s| = \sqrt{s_1^2 + s_2^2}$ and $K \equiv (\vec{k}, i\omega_n)$.

As SOC and magnetic field are weak, the interaction of two quasiparticles with momenta \vec{k} and \vec{k}' can be described by an $SU(2)$ -invariant Landau function $\hat{F}(\vartheta) = \hat{\sigma}_0 \hat{\sigma}'_0 F^s(\vartheta) + \hat{\vec{\sigma}} \cdot \hat{\vec{\sigma}}' F^a(\vartheta)$, where $\vartheta = \theta_{\vec{k}} - \theta_{\vec{k}'}$. Within this approximation, the charge and spin collective modes are decoupled, and we focus on the interaction in the spin channel, parameterized by $F^a(\vartheta)$.

To set the stage, we discuss the SL mode in the absence of SOC. Kohn's theorem protects the $q = 0$ term in the dispersion from renormalization by eei [23, 24]; therefore, $\Omega(\vec{0}, \vec{B}) = \Delta_Z$. The absence of the linear-in- q term is guaranteed by symmetry: the only rotationally-invariant combination of \vec{q} and \vec{B} is $\vec{q} \cdot \vec{B}$, which is a pseudoscalar and therefore not allowed since Ω must be a scalar. For the quadratic term, although $(\vec{q} \cdot \vec{B})^2$ is allowed by symmetry, such a term is absent because \vec{q} is the orbital momentum whereas B acts only on electron spins. The energy of a particle-hole pair formed by fermions with momenta \vec{k} and $\vec{k} + \vec{q}$ and with opposite spins is $\varepsilon_{\vec{k}+\vec{q},\pm} - \varepsilon_{\vec{k},\mp} = v_F q \cos(\theta_{\vec{k}} - \theta_{\vec{q}}) + q^2/2m \pm \Delta_Z$. Integration over $\theta_{\vec{k}}$ eliminates the dependence on $\theta_{\vec{q}}$, hence only a q^2 term is possible. Combining the symmetry arguments with dimensional analysis, we find

$$\Omega(\vec{q}, \vec{B}) = \Delta_Z + a_2(\{F^a\}) \frac{(v_F q)^2}{\Delta_Z}, \quad (7)$$

where a_2 is a dimensionless function which depends on the angular harmonics of $F^a(\vartheta)$: $a_2(\{F^a\}) = a_2(F_0^a, F_1^a, F_2^a, \dots)$. The precise form of a_2 is determined by a microscopic theory [15, 25].

We now apply the same reasoning to CSWs. If only Rashba SOC is present, there is no preferred in-plane direction, hence a linear-in- q term is absent while the quadratic term is isotropic. Therefore, the dispersions of the three CSWs can be written as $\Omega_\alpha(\vec{q}, \vec{0}) = \tilde{a}_0^\alpha(\{F^a\})\Delta_R + \tilde{a}_2^\alpha(\{F^a\})(v_F q)^2/\Delta_R$, where $\tilde{a}_{0,2}^\alpha$ are some other dimensionless functions of the FL parameters and $\alpha = 1 \dots 3$. Since Kohn's theorem does not apply in the presence of SOC systems, $\tilde{a}_0^\alpha \neq 1$. Explicit forms of the functions $\tilde{a}_{0,2}^\alpha$ were found in Refs. [11–13]. If only Dresselhaus SOC is present, the Hamiltonian can be transformed to the Rashba form by replacing $\theta_{\vec{k}} \rightarrow \pi/2 - \theta_{\vec{k}}$. Therefore, the CSWs are the same in both cases although the Hamiltonians have different symmetries.

If both Rashba and Dresselhaus types of SOC are present, the $q = 0$ term in the dispersion is obviously isotropic, while a linear-in- q term is not allowed by symmetry. Indeed, since the dispersion must be analytic in q , the linear term could only be of the form $c_1 q_1 + c_2 q_2$ with $c_{1,2} = \text{const}$, but such a form does not obey the symmetries of the D_{2d} group (rotation by π and reflection about the diagonal plane). The quadratic term can have an anisotropic part of the form $q_1 q_2 \propto \sin 2\theta_{\vec{q}}$, but the prefactor of such a combination should be proportional to the product $\Delta_R \Delta_D$, and thus the anisotropic part is small compared to the isotropic, q^2 part.

Finally, let both types of SOC and the magnetic field be present but, in agreement with the experimental conditions of Refs. [7–9], $\Delta_Z \gg \Delta_R, \Delta_D$. The $q = 0$ term in the dispersion then depends on the orientation of \vec{B} in the plane; the D_{2d} -symmetry forces this dependence to be of $\sin 2\theta_{\vec{B}}$ form. Since the anisotropic part of $\Omega(\vec{0}, \theta_{\vec{B}})$ is non-zero only if both types of SOC are present, the

coefficient of the $\sin 2\theta_{\vec{B}}$ term must be on the order of $\Delta_R \Delta_D / \Delta_Z$. In addition to the anisotropic part, there are also isotropic corrections of order Δ_R^2 / Δ_Z and Δ_D^2 / Δ_Z .

To lowest order in SOC, the form of the linear term is determined by the symmetries of the Rashba (group $C_{\infty v}$) and Dresselhaus (group D_{2d}) types of SOC. In both cases, we need to form a scalar (Ω) out of a polar vector (\vec{q}) and a pseudovector (\vec{B}). In the $C_{\infty v}$ group, this is only possible by forming the Rashba invariant $B_1 q_2 - B_2 q_1 \propto \sin(\theta_{\vec{q}} - \theta_{\vec{B}})$, which is the same term as in the original Rashba Hamiltonian with $\hat{\sigma} \rightarrow \vec{B}$. Likewise, the only possible scalar in the D_{2d} group is the Dresselhaus invariant $B_1 q_1 - B_2 q_2 \propto \cos(\theta_{\vec{q}} + \theta_{\vec{B}})$. The quadratic term in the high-field limit can be taken the same as the quadratic term in the SL mode, Eq. (7) [26].

Combining together all the arguments given above, we arrive at the following form of the coefficients in Eq. (1)

$$\begin{aligned} \Omega(\vec{0}, \vec{B}) &= \Delta_Z + a_0(\{F^a\}) \frac{\Delta_R^2 + \Delta_D^2}{\Delta_Z} \\ &\quad + \tilde{a}_0(\{F^a\}) \frac{\Delta_R \Delta_D}{\Delta_Z} \sin 2\theta_{\vec{B}}, \\ w(\theta_{\vec{q}}, \theta_{\vec{B}}) &= v_F a_1(\{F^a\}) \left[\frac{\Delta_R}{\Delta_Z} \sin(\theta_{\vec{q}} - \theta_{\vec{B}}) \right. \\ &\quad \left. + \frac{\Delta_D}{\Delta_Z} \cos(\theta_{\vec{q}} + \theta_{\vec{B}}) \right], \\ A &= a_2(\{F^a\}) \frac{v_F^2}{\Delta_Z}, \end{aligned} \quad (8)$$

where a_0, \tilde{a}_0, a_1 and a_2 are dimensionless functions.

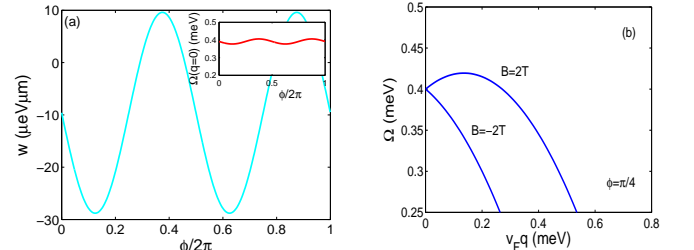


FIG. 3: a) Variation of the CSW velocity w [Eq. (1)] with the angle $\phi = \theta_{\vec{B}} - \pi/2$, as defined in Refs. [7–9]. Inset: variation of the frequency at $q = 0$ with ϕ . b) Dispersion of a CSW in the CdMnTe quantum well in a $2T$ field ($\vec{B} \perp \vec{q}$) for $\phi = \pi/4$. The negative sign of the field implies flipping its direction.

We confirm Eq. (8) by an explicit calculation within the s -wave approximation, in which the Landau function contains only the zeroth harmonic: $F^a(\theta) = F_0^a$. In this case, the FL theory is equivalent to the Random Phase Approximation (RPA) in the spin channel [13, 14], and we adopt the RPA method for convenience. Summing up the series of ladder diagrams, we obtain for the tensor of

the spin susceptibility

$$\hat{\chi}(Q) = \frac{(g\mu_B)^2}{4} \hat{\chi}^1(Q) \left[\hat{I} + \frac{F_0^a}{2} \hat{\chi}^1(Q) \right]^{-1}, \quad (9)$$

where $Q = (\vec{q}, i\Omega_n)$, g is the effective Landé factor, μ_B is the Bohr magneton, \hat{I} is the 3×3 identity matrix, $\chi_{ij}^1(Q) = -\int_K \text{Tr} [\hat{\sigma}_i \hat{G}_K^1 \hat{\sigma}_j \hat{G}_{K+Q}^1]$, and G_K^1 differs from G_K^0 in Eq. (6a) only in that the bare Zeeman energy is replaced by the renormalized one, $\Delta_Z^* = \Delta_Z/(1 + F_0^a)$, while Δ_R and Δ_D are not renormalized in the s -wave approximation [10, 27, 28]. The collective modes are the roots of the equation

$$\text{Det} \left[\hat{I} + \frac{F_0^a}{2} \hat{\chi}^1(\vec{q}, i\Omega_n \rightarrow \Omega + i0^+) \right] = 0. \quad (10)$$

The details on solving this equation are given in the Supplementary Material (SM). Here, we only mention that the results coincide with those in Eq. (8), and explicit expressions for the dimensionless functions of the FL parameter F_0^a read

$$\begin{aligned} \tilde{a}_0(F_0^a) &= -\frac{(1 + F_0^a)(2 + F_0^a)}{2F_0^a}, \\ a_0(F_0^a) &= \frac{(1 + F_0^a)(2 + 3F_0^a)}{4F_0^a}, \\ a_1(F_0^a) &= -\frac{(1 + F_0^a)^2 [(4 + F_0^a)(1 + F_0^a) + (F_0^a)^2]}{F_0^a(2 + F_0^a)^2}, \\ a_2(F_0^a) &= \frac{(1 + F_0^a)^2}{2F_0^a}. \end{aligned} \quad (11)$$

[The form of a_2 and a_0 coincides with the previous results [14, 25] in the s -wave approximation.] For repulsive eei , $F_0^a < 0$ and the quadratic term in the dispersion is always negative, while the sign of the interaction-dependent prefactor of the linear term [$a_1(F_0)$] is positive. However, the overall sign of the linear term depends on the signs of the Rashba and Dresselhaus couplings.

The continuum of SFE corresponds to interband transitions and is given by a set of Ω that satisfy $\Omega = |\varepsilon_{\vec{k},+} - \varepsilon_{\vec{k},-}|$ for all states on the FS. Because ε_{\pm} vary around the FS, Ω varies between the minimum (Ω_{\min}) and maximum (Ω_{\max}) values, which determine a finite width of the continuum even at $q = 0$ (see Fig. 1a). In the presence of the \vec{B} , both Ω_1 and Ω_2 depend on $\theta_{\vec{B}}$. The anisotropic part of $\varepsilon_{\vec{k},\pm}$ is given by $\Delta_R \Delta_D \sin 2\theta_{\vec{k}} + \Delta_R \Delta_Z^* \sin(\theta_{\vec{k}} - \theta_{\vec{B}}) + \Delta_D \Delta_Z^* \cos(\theta_{\vec{k}} + \theta_{\vec{B}})$. As one can see, changing $\theta_{\vec{B}} \rightarrow \theta_{\vec{B}} + \pi$ can be compensated by $\theta_{\vec{k}} \rightarrow \theta_{\vec{k}} + \pi$, and thus $\Omega_{\min, \max}(\theta_{\vec{B}})$ have a period of π . Figure 1b shows $\Omega_{\min, \max}(\theta_{\vec{B}})$ for a range of the magnetic field.

We are now in a position to apply our results to recent Raman data on $\text{Cd}_{1-x}\text{Mn}_x\text{Te}$ quantum well [7–9]. In these experiments, \vec{q} and \vec{B} are chosen to be perpendicular to each other, i.e., $\theta_{\vec{B}} - \theta_{\vec{q}} = \pm\pi/2$. Accordingly,

the dispersion in Eqs. (1) and (8) is simplified to

$$\begin{aligned} \Omega(\vec{q}, \vec{B}) &= \Delta_Z [1 + a_0(r^2 + d^2) + \tilde{a}_0 r d \sin 2\theta_{\vec{B}} \\ &\quad \pm a_1 (r - d \sin 2\theta_{\vec{B}}) \frac{v_F q}{\Delta_Z} + a_2 \frac{(v_F q)^2}{\Delta_Z}]. \end{aligned} \quad (12)$$

where $r = \Delta_R/\Delta_Z$ and $d = \Delta_D/\Delta_Z$. Up to the angular dependence of the mode mass, this form was conjectured in Refs. [8, 9] on phenomenological grounds. The measured frequency of the mode at $q = 0$ gives $\Delta_Z = 0.4 \text{ meV}$ at $B = 2 \text{ T}$. For the number density of $n = 2.7 \times 10^{11} \text{ cm}^{-2}$ and effective mass of $m^* = 0.1 m_e$ (m_e is the bare electron mass), $k_F = 1.3 \times 10^{-2} \text{ \AA}^{-1}$ and $v_F = 1.0 \text{ eV} \cdot \text{\AA}$. The range of $v_F q$ is $0 - 0.6 \text{ meV}$. Using these parameters, we achieve the best agreement with the data for $\Omega(\vec{0}, \vec{B})$ and $w(\pm\pi/2 + \theta_{\vec{B}}, \theta_{\vec{B}})$ (Fig. 3a) by choosing $F_0^a = -0.41$, $\Delta_R \approx 0.05 \text{ meV}$, and $\Delta_D \approx 0.1 \text{ meV}$ which corresponds to the Rashba and Dresselhaus coupling constants of $\alpha = 1.9 \text{ meV} \cdot \text{\AA}$ and $\beta = 3.8 \text{ meV} \cdot \text{\AA}$, in agreement with the values found in Ref. [9]. The screened Coulomb interaction with a dielectric constant of 10.0 for a CdTe quantum well yields $F_0^a \approx -0.35$. The q -dependence of the mode (Fig. 3b) reproduces the experimentally observed one very well. Based on this agreement between the theory and experiment, we argue that the collective mode observed in Refs. [7–9] is, in fact, one of the sought-after CSWs [10–13], probed in the regime of a strong magnetic field. The same type of experiments performed at lower fields on systems with stronger SOC and/or higher mobilities, e.g., on InGaAs/InAlAs, should reveal the whole spectrum shown in Fig. 1a.

The phenomenological model of Refs. [8, 9] describes the data assuming very strong (up to a factor of 6.5) renormalization of SOC by many-body effects, which is not consistent with the moderate (< 2) values of parameter r_s in $\text{Cd}_{1-x}\text{Mn}_x\text{Te}$. Our theory explains the data with no such assumption. References [8] and [9] argued that the dispersion can be obtained by a linear shift of the SL dispersion: $Aq^2 \rightarrow A|\vec{q} + \vec{q}_0|^2$. As is evident from Eq. (11), however, this form holds only at weak coupling ($|F_0^a| \ll 1$).

In conclusion, we developed a microscopic theory of Raman scattering by a 2D FL in the presence of both Rashba and Dresselhaus SOC, and subject to an in-plane magnetic field. An interplay between eei and SOC leads to resonance peaks at frequencies of CSWs. All the features of one of those peaks observed recently on a CdTe quantum well are successfully accounted for by this theory. The formalism developed here can be readily extended to other 2D systems with broken inversion symmetry, such as graphene on transition-metal-dichalcogenide substrates and surface states of topological insulators/superconductors.

The authors are grateful to Y. Gallais, A. Kumar, I. Paul, F. Perez, and C. A. Ullrich for stimulating discus-

sions and to M. Imran for his help at the initial stage of this work. *Note:* After this manuscript was almost completed, we learned of a recent preprint, Ref. [29], which also identifies a $2\theta_{\vec{B}}$ modulation of the mode frequency at $q = 0$ as a second-order effect in SOC.

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SUPPLEMENTARY MATERIAL

RAMAN RESPONSE

The Raman scattering cross-section for a non-interacting two-dimensional electron gas (2DEG) in cross-polarized geometry is given by [6]

$$\frac{d^2\mathcal{A}}{d\Omega d\mathcal{O}} \propto \sum_{\mu\mu'} |\gamma_{\mu\mu'}|^2 \text{Im} L_{\mu\mu'}^0 - \frac{2\pi e^2}{q} \text{Im} \frac{Z\bar{Z}}{\epsilon(q, \Omega)}, \quad (13a)$$

$$\gamma_{\mu\mu'} = i\langle \vec{n} \cdot \hat{\sigma} e^{i\vec{q}\cdot\vec{r}} \rangle_{\mu\mu'}, \quad (13b)$$

$$L_{\mu\mu'}^0 = \frac{f(\varepsilon_\mu) - f(\varepsilon_{\mu'})}{\Omega + \varepsilon_\mu - \varepsilon_{\mu'} + i\delta}, \quad (13c)$$

$$Z = \sum_{\mu\mu'} \gamma_{\mu\mu'} \langle e^{-i\vec{q}\cdot\vec{r}} \rangle_{\mu\mu'} L_{\mu\mu'}^0; \quad \bar{Z} = \sum_{\mu\mu'} \gamma_{\mu\mu'}^* \langle e^{i\vec{q}\cdot\vec{r}} \rangle_{\mu\mu'} L_{\mu\mu'}^0 \quad (13d)$$

$$\epsilon(q, \Omega) = \epsilon + \frac{2\pi e^2}{q} \sum_{\mu\mu'} |\langle e^{i\vec{q}\cdot\vec{r}} \rangle_{\mu\mu'}|^2 L_{\mu\mu'}^0. \quad (13e)$$

Here, ϵ is the background dielectric constant, μ and μ' refer to the quantum numbers of electrons which include the momentum and spin/chirality: $\mu = \{\vec{k}, \nu\}$, $\mu' = \{\vec{k}', \nu'\}$ with $\nu, \nu' = \pm 1$, ε_μ is the energy of a state with quantum number μ , $f(\varepsilon)$ is the Fermi function, and $\langle X \rangle_{\mu\mu'} \equiv \int d\vec{r} \psi_\mu^\dagger(\vec{r}) X(\vec{r}) \psi_{\mu'}(\vec{r})$ is the matrix element of $X(\vec{r})$ between states μ and μ' . Furthermore, $\vec{n} = \vec{e}_I \times \vec{e}_S$, where $\vec{e}_{I,S}$ are the polarizations of incident and scattered light, correspondingly. The two terms in Eq. (13a) refer to two contributions in Fig. 2 of the Main Text (MT): the first term is the spin part (given by the first diagram) while the second term is the mixed spin-charge part (given by the sum of diagrams starting from the second one). The matrix elements are computed with respect to the eigenvectors of Hamiltonian in Eq. (4) of the MT

$$\psi_\mu(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \begin{pmatrix} i\nu e^{-i\phi_{\vec{k}}} \\ 1 \end{pmatrix}, \quad (14)$$

where $\tan \phi_{\vec{k}} = -s_1/s_2$ and $s_{1,2}$ are defined by Eq. (5) of the MT. Explicitly, we obtain:

$$\begin{aligned} \gamma_{\mu\mu'} &= i\delta_{\vec{k}-\vec{k}'-\vec{q}} \left[n_1 \left\{ i\nu' e^{-i\phi_{\vec{k}+\vec{q}}} - i\nu e^{i\phi_{\vec{k}}} \right\} + n_2 \left\{ -\nu' e^{-i\phi_{\vec{k}+\vec{q}}} - \nu e^{i\phi_{\vec{k}}} \right\} + n_3 \left\{ \nu\nu' e^{-i(\phi_{\vec{k}+\vec{q}}-\phi_{\vec{k}})} - 1 \right\} \right], \\ \langle e^{i\vec{q}\cdot\vec{r}} \rangle_{\mu\mu'} &= \delta_{\vec{k}-\vec{k}'-\vec{q}} \left[\nu\nu' e^{-i(\phi_{\vec{k}+\vec{q}}-\phi_{\vec{k}})} + 1 \right]. \end{aligned} \quad (15)$$

We are interested in the small- q limit, when the q -dependent terms in the dispersions of chiral spin waves (CSWs) are corrections to the frequencies of these waves at $q = 0$, which implies that $v_F q \ll \Omega$ with v_F being the Fermi velocity in absence of both the magnetic field and spin-orbit coupling (SOC). At the same time, both the magnetic field and SOC are assumed to be weak: $\Omega \ll v_F k_F$. Combining the two inequalities above, we obtain

$$q \ll \Omega/v_F \ll k_F. \quad (16)$$

The first part of this inequality ensures that the diagonal parts of $L_{\mu\mu'}^0$ in Eq. (13a) are small by charge conservation: $L_{\vec{k}, \pm; \vec{k}+\vec{q}, \pm}^0 \propto q$. Therefore, the main contribution to the cross-section in this limit comes from off-diagonal terms with $\nu\nu' = -1$, which correspond to processes that flip spin/chirality. Furthermore, Eq. (16) also implies that $q \ll k_F$; therefore, $\phi_{\vec{k}+\vec{q}}$ can be approximated by $\phi_{\vec{k}}$, upon which the off-diagonal matrix elements are simplified to

$$\gamma_{\mu\mu'} \approx -2i\delta_{\vec{k}-\vec{k}'-\vec{q}} [in_1\nu \cos \phi_{\vec{k}} + in_2\nu \sin \phi_{\vec{k}} + n_3], \quad (17a)$$

$$|\gamma_{\mu\mu'}|^2 \approx 4\delta_{\vec{k}-\vec{k}'-\vec{q}} [(n_1 \cos \phi_{\vec{k}} + n_2 \sin \phi_{\vec{k}})^2 + n_3^2], \quad (17b)$$

$$\langle e^{i\vec{q}\cdot\vec{r}} \rangle_{\mu\mu'} \approx \nu\nu' + 1 = 0. \quad (17c)$$

By the same argument, the diagonal terms in Z and \bar{Z} in Eq. (13a) are small as q , while the off-diagonal components of $\langle e^{i\vec{q}\cdot\vec{r}} \rangle_{\mu\mu'}$ are small by Eq. (17c). Therefore, the second term in Eq. (13a) can be neglected compared to the first one, and the cross-section is reduced to

$$\frac{d^2\mathcal{A}}{d\Omega d\mathcal{O}} \propto \int \frac{d^2k}{(2\pi)^2} [n_1^2 \cos^2 \phi_{\vec{k}} + n_2^2 \sin^2 \phi_{\vec{k}} + n_1 n_2 \sin 2\phi_{\vec{k}} + n_3^2] \int_{\omega} (g_+ \tilde{g}_- + g_- \tilde{g}_+), \quad (18)$$

where $g_\nu = 1/(i\omega - \varepsilon_{\vec{k},\nu})$ and $\tilde{g}_\nu = 1/(i\{\omega + \Omega\} - \varepsilon_{\vec{k}+\vec{q},\nu})$. The various terms in the equation above can be expressed via the components of the spin-spin correlation function in the chiral basis:

$$\begin{aligned}\chi_{11}^0(Q) &= - \int_{\vec{k}} \int_{\omega} (g_+ \tilde{g}_- + g_- \tilde{g}_+) \cos^2 \phi_{\vec{k}}, \\ \chi_{22}^0(Q) &= - \int_{\vec{k}} \int_{\omega} (g_+ \tilde{g}_- + g_- \tilde{g}_+) \sin^2 \phi_{\vec{k}}, \\ \chi_{12}^0(Q) &= - \int_{\vec{k}} \int_{\omega} (g_+ \tilde{g}_- + g_- \tilde{g}_+) \sin \phi_{\vec{k}} \cos \phi_{\vec{k}}, \\ \chi_{21}^0(Q) &= \chi_{12}^0(Q), \\ \chi_{33}^0(Q) &= - \int_{\vec{k}} \int_{\omega} (g_+ \tilde{g}_- + g_- \tilde{g}_+),\end{aligned}\tag{19}$$

where $\int_{\vec{k}} \equiv \int d^2k/(2\pi)^2$ and $\int_{\omega} \equiv \int d\omega/2\pi$. Making use of the definitions, we re-write Eq. (18) as

$$\frac{d^2 \mathcal{A}}{d\Omega d\mathcal{O}} \propto n_1^2 \chi_{11}^0 + n_2^2 \chi_{22}^0 + n_1 n_2 (\chi_{12}^0 + \chi_{21}^0) + n_3^2 \chi_{33}^0,\tag{20}$$

which is the result given in Eq. (3) of the MT.

Generalization of Eq. (20) for the case of interacting electrons amounts simply to replacing the free-electron spin-spin correlation function by the interacting one.

DISPERSION OF SPIN CHIRAL WAVES

The general form of the dispersion is established in Eq. 8 of the MT using symmetry and dimensional analysis. Here, we confirm this form by an explicit calculation which yields the forms of functions a_0 , \tilde{a}_0 , a_1 , and a_2 . In the Random Phase Approximation (RPA) or, equivalently, in the s -wave approximation for the Landau interaction function, the eigenmode equation is given by Eq. (10) the MT. Therefore, the problem reduces to finding expansions of the spin-spin correlation functions, $\chi_{ab}^1(Q)$, to order q^2 [as in the main text, $Q \equiv (\vec{q}, i\Omega_n)$]. Unless stated otherwise, all frequencies in the Supplementary Material are the Matsubara ones. Whenever it does not lead to a confusion, the Matsubara index n will be suppressed and the frequencies will be denoted simply as $i\Omega$. Analytic continuation to real frequencies will be done at the final step. We follow the general scheme developed in Refs. [13, 14] to find the dispersions of the collective modes. For brevity, we will switch to notations where $F_0^a = -u$ and $\chi_{ab}^1 = -\Pi_{ab}$.

General properties of the polarization bubble

The polarization bubble is defined as

$$\Pi_{ab}(Q) = - \int_{\vec{k}} \int_{\omega} \text{Tr} \left[\hat{\sigma}_a \hat{G}_{K+Q}^1 \hat{\sigma}_b \hat{G}_K^1 \right],\tag{21}$$

where $a, b \in \{0 \dots 3\}$ and the Green's function G_P^1 is obtained from G_P^0 in Eq. (6a) of the MT by replacing $\Delta_Z \rightarrow \Delta_Z^* = \Delta_Z/(1-u)$. This yields

$$\begin{aligned}\Pi_{ab}(Q) &= \int_{\vec{k}} \int_{\omega} \left[\frac{g_+ \tilde{g}_+ + g_- \tilde{g}_-}{4} \left\{ 2\delta_{ab} + \text{Tr}[\hat{\sigma}_a \hat{\sigma}_i \hat{\sigma}_b \hat{\sigma}_j] \frac{s_i \tilde{s}_j}{|s| |\tilde{s}|} \right\} + \frac{g_+ \tilde{g}_- + g_- \tilde{g}_+}{4} \left\{ 2\delta_{ab} - \text{Tr}[\hat{\sigma}_a \hat{\sigma}_i \hat{\sigma}_b \hat{\sigma}_j] \frac{s_i \tilde{s}_j}{|s| |\tilde{s}|} \right\} \right. \\ &\quad \left. + \frac{g_+ \tilde{g}_+ - g_- \tilde{g}_-}{4} \left\{ \text{Tr}[\hat{\sigma}_b \hat{\sigma}_a \hat{\sigma}_j] \frac{s_j}{|s|} + \text{Tr}[\hat{\sigma}_a \hat{\sigma}_b \hat{\sigma}_j] \frac{\tilde{s}_j}{|\tilde{s}|} \right\} + \frac{g_+ \tilde{g}_- - g_- \tilde{g}_+}{4} \left\{ \text{Tr}[\hat{\sigma}_b \hat{\sigma}_a \hat{\sigma}_j] \frac{s_j}{|s|} - \text{Tr}[\hat{\sigma}_a \hat{\sigma}_b \hat{\sigma}_j] \frac{\tilde{s}_j}{|\tilde{s}|} \right\} \right],\end{aligned}\tag{22}$$

where $i, j \in \{1, 2\}$ and a tilde above any quantity means that its $(2+1)$ momentum is shifted by Q with respect to the momentum of the corresponding quantity without a tilde. Because the magnetic field and SOC are weak, integration over the actual (anisotropic) Fermi surface can be replaced by that over the circular Fermi surface of radius k_F . Accordingly,

$$\int_{\vec{k}} \dots = N_F \int \frac{d\theta}{2\pi} \int d\xi_{\vec{k}} \dots \equiv N_F \int_{\theta} \int_{\xi_{\vec{k}}} \dots\tag{23}$$

where $N_F = k_F/2\pi v_F$ is the 2D density of states and $\xi_{\vec{k}} = v_F(k - k_F)$. In the same approximation, $\varepsilon_{\vec{k},\nu} = \xi_{\vec{k}} + \nu|s|$, where $|s| = \sqrt{s_1^2 + s_2^2}$ is evaluated at $\xi = 0$ but does depend on the azimuthal angle θ of \vec{k} . Furthermore, we will be neglecting the difference between $|s|$ at $\vec{k} + \vec{q}$ and $|s|$ at \vec{k} : keeping such terms would amount to higher order corrections. Accordingly, we approximate $\varepsilon_{\vec{k}+\vec{q},\nu} = \xi_{\vec{k}+\vec{q}} + \nu|\tilde{s}| \approx \xi_{\vec{k}} + \vec{v}_F \cdot \vec{q} + \nu|s|$, where we also neglected the $\mathcal{O}(q^2)$ term as being higher order in q/k_F .

The chiral Green's functions at momentum $\vec{k} + \vec{q}$ need to be expanded to second order in \vec{q} . Within the same approximations as specified above,

$$\tilde{g}_\nu \equiv g_\nu(\vec{k} + \vec{q}, i\omega + i\Omega) = \bar{g}_\nu + \partial_j \bar{g}_\nu q_j + \frac{1}{2} \partial_j \partial_{j'} \bar{g}_\nu q_j q_{j'} \approx \bar{g}_\nu + \bar{g}_\nu^2 \vec{v}_F \cdot \vec{q} + \bar{g}_\nu^3 (\vec{v}_F \cdot \vec{q})^2, \quad (24)$$

where $\bar{g}_\nu \equiv g_\nu(\vec{k}, i\omega + i\Omega)$ and $\partial_l \equiv \partial/\partial k_l$

We will need the following integrals

$$\begin{aligned} \int_{\xi_{\vec{k}}} \int_{\omega} g_\nu \bar{g}_{\nu'} &= -\frac{(\nu - \nu')|s|}{i\Omega + (\nu - \nu')|s|}, \\ \int_{\xi_{\vec{k}}} \int_{\omega} g_\nu \bar{g}_{\nu'}^2 &= \frac{i\Omega}{(i\Omega + (\nu - \nu')|s|)^2}, \\ \int_{\xi_{\vec{k}}} \int_{\omega} g_\nu \bar{g}_{\nu'}^3 &= \frac{i\Omega}{(i\Omega + (\nu - \nu')|s|)^3}, \end{aligned} \quad (25)$$

where $\int_{d\xi_{\vec{k}}} \equiv \int d\xi_{\vec{k}}$ and $\int_{\omega} \equiv \int d\omega/(2\pi)$. Using these integrals, we find

$$\int_{\xi_{\vec{k}}} \int_{\omega} g_+ \tilde{g}_+ = \frac{\vec{v}_F \cdot \vec{q}}{i\Omega} + \frac{(\vec{v}_F \cdot \vec{q})^2}{(i\Omega)^2} \quad (26a)$$

$$\int_{\xi_{\vec{k}}} \int_{\omega} g_- \tilde{g}_- = \int_{\xi_{\vec{k}}} \int_{\omega} g_+ \tilde{g}_+, \quad (26b)$$

$$\int_{\xi_{\vec{k}}} \int_{\omega} g_{\pm} \tilde{g}_{\mp} = \mp \frac{2|s|}{i\Omega \pm 2|s|} + \frac{i\Omega}{(i\Omega \pm 2|s|)^2} \vec{v}_F \cdot \vec{q} + \frac{i\Omega}{(i\Omega \pm 2|s|)^3} (\vec{v}_F \cdot \vec{q})^2. \quad (26c)$$

When $\tilde{s}_a \approx s_a$ in Eq. 22, the spin-charge components of the polarization operator, $\Pi_{a0}(Q)$ with $a \neq 0$, reduce to a combination $\Pi_{a0} \propto \int_{\xi_{\vec{k}}} \int_{\omega} (g_+ \tilde{g}_+ - g_- \tilde{g}_-) s_a$ which is equal to zero by Eq. 26b. Thus the spin sector is decoupled from the charge sector in the limit of $q/k_F \rightarrow 0$. Restricting $a, b \in \{1, 2, 3\}$, we obtain

$$\begin{aligned} \text{Tr}[\hat{\sigma}_a \hat{\sigma}_b \hat{\sigma}_j] s_j &= i\lambda_{abj} s_j \\ \text{Tr}[\hat{\sigma}_a \hat{\sigma}_i \hat{\sigma}_b \hat{\sigma}_j] s_i s_j &= -2\delta_{ab} |s|^2 + 4s_a s_b, \end{aligned} \quad (27)$$

where λ_{abc} is Levi-Civita tensor and it is understood that $s_3 = 0$. We thus obtain a compact form of the spin-spin correlation function:

$$\Pi_{ab}(Q) = \frac{k_F}{2\pi v_F} \int_{\theta} \int_{\xi_{\vec{k}}} \int_{\omega} \left[(g_+ \tilde{g}_+ + g_- \tilde{g}_-) \frac{s_a s_b}{|s|^2} + (g_+ \tilde{g}_- + g_- \tilde{g}_+) \left(\delta_{ab} - \frac{s_a s_b}{|s|^2} \right) + i(g_+ \tilde{g}_- - g_- \tilde{g}_+) \lambda_{bac} \frac{s_c}{|s|} \right], \quad (28)$$

where the integrals over ω and $\xi_{\vec{k}}$ are to be substituted from Eqs. (26a-26b). For further convenience, we also list explicit formulas for s_a and related quantities:

$$\begin{aligned} s_1 &= \frac{1}{2} (\Delta_R \sin \theta + \Delta_D \cos \theta - \Delta_Z^* \cos \theta_{\vec{B}}), \\ s_2 &= \frac{1}{2} (-\Delta_R \cos \theta - \Delta_D \sin \theta - \Delta_Z^* \sin \theta_{\vec{B}}), \\ 4s_1^2 &= \Delta_R^2 \sin^2 \theta + \Delta_D^2 \cos^2 \theta + (\Delta_Z^*)^2 \cos^2 \theta_{\vec{B}} + \Delta_R \Delta_D \sin 2\theta - 2\Delta_R \Delta_Z^* \sin \theta \cos \theta_{\vec{B}} - 2\Delta_D \Delta_Z^* \cos \theta \cos \theta_{\vec{B}}, \\ 4s_2^2 &= \Delta_R^2 \cos^2 \theta + \Delta_D^2 \sin^2 \theta + (\Delta_Z^*)^2 \sin^2 \theta_{\vec{B}} + \Delta_R \Delta_D \sin 2\theta + 2\Delta_R \Delta_Z^* \cos \theta \sin \theta_{\vec{B}} + 2\Delta_D \Delta_Z^* \sin \theta \sin \theta_{\vec{B}}, \\ 4|s|^2 &= \Delta_R^2 + \Delta_D^2 + (\Delta_Z^*)^2 + 2\Delta_R \Delta_D \sin 2\theta - 2\Delta_R \Delta_Z^* \sin(\theta - \theta_{\vec{B}}) - 2\Delta_D \Delta_Z^* \cos(\theta + \theta_{\vec{B}}), \\ 4s_1 s_2 &= -(\Delta_R^2 + \Delta_D^2) \sin \theta \cos \theta + (\Delta_Z^*)^2 \sin \theta_{\vec{B}} \cos \theta_{\vec{B}} - \Delta_R \Delta_D + \Delta_R \Delta_Z^* \cos(\theta + \theta_{\vec{B}}) + \Delta_D \Delta_Z^* \sin(\theta - \theta_{\vec{B}}), \end{aligned} \quad (29)$$

where $\theta_{\vec{B}}$ is the angle between \vec{B} and the x_1 axis.

We can now apply the general result, Eq. (28) to particular situations. In what follows, we will make $v_F q$ and Ω dimensionless by rescaling to $\Delta_Z^* = \Delta_Z/(1-u)$, and define $r \equiv \Delta_R/\Delta_Z^*$ and $d \equiv \Delta_D/\Delta_Z^*$. Note that these definitions differ from those in the MT, where Δ_R and Δ_D are rescaled to Δ_Z .

Silin-Leggett mode

To test our general formula, we apply it first to a simple case of the Silin-Leggett mode, the dispersion of which is known [25]. The Silin-Leggett mode is the collective mode in the absence of SOC ($r = d = 0$) and in the presence of \vec{B} , which we assume to be along the x_1 axis. In this case, $|s| = \Delta_Z^*$ is independent of angle θ . Using Eq. (28), we find:

$$\begin{aligned}\frac{\Pi_{11}(Q)}{-2N_F} &= \frac{(v_F q)^2}{2\Omega^2} + \frac{\sin^2 \theta_{\vec{B}}}{\Omega^2 + 1} \left(1 - \frac{6\Omega^4 + 3\Omega^2 + 1}{\Omega^2(\Omega^2 + 1)^2} \frac{(v_F q)^2}{2} \right), \\ \frac{\Pi_{22}(Q)}{-2N_F} &= \frac{(v_F q)^2}{2\Omega^2} + \frac{\cos^2 \theta_{\vec{B}}}{\Omega^2 + 1} \left(1 - \frac{6\Omega^4 + 3\Omega^2 + 1}{\Omega^2(\Omega^2 + 1)^2} \frac{(v_F q)^2}{2} \right), \\ \frac{\Pi_{33}(Q)}{-2N_F} &= \frac{1}{\Omega^2 + 1} \left(1 + \frac{\Omega^6 - 3\Omega^4}{\Omega^2(\Omega^2 + 1)^2} \frac{(v_F q)^2}{2} \right), \\ \frac{\Pi_{12}(Q)}{-2N_F} &= -\frac{\sin \theta_{\vec{B}} \cos \theta_{\vec{B}}}{\Omega^2 + 1} \left(1 - \frac{6\Omega^4 + 3\Omega^2 + 1}{\Omega^2(\Omega^2 + 1)^2} \frac{(v_F q)^2}{2} \right), \\ \frac{\Pi_{13}(Q)}{-2N_F} &= -\frac{\Omega \sin \theta_{\vec{B}}}{\Omega^2 + 1} \left(1 - \frac{3\Omega^2 - 1}{(\Omega^2 + 1)^2} \frac{(v_F q)^2}{2} \right), \\ \frac{\Pi_{23}(Q)}{-2N_F} &= \frac{\Omega \cos \theta_{\vec{B}}}{\Omega^2 + 1} \left(1 - \frac{3\Omega^2 + 1}{(\Omega^2 + 1)^2} \frac{(v_F q)^2}{2} \right).\end{aligned}\quad (30)$$

Π_{ab} can be written more compactly in a matrix form as

$$\frac{\hat{\Pi}(Q)}{-2N_F} = \begin{pmatrix} \kappa_a + (\mathcal{B} + \kappa_b) \sin^2 \theta_{\vec{B}} & -(\mathcal{B} + \kappa_b) \sin \theta_{\vec{B}} \cos \theta_{\vec{B}} & (\mathcal{C} + \kappa_c) \sin \theta_{\vec{B}} \\ -(\mathcal{B} + \kappa_b) \sin \theta_{\vec{B}} \cos \theta_{\vec{B}} & \kappa_a + (\mathcal{B} + \kappa_b) \cos^2 \theta_{\vec{B}} & -(\mathcal{C} + \kappa_c) \cos \theta_{\vec{B}} \\ -(\mathcal{C} + \kappa_c) \sin \theta_{\vec{B}} & (\mathcal{C} + \kappa_c) \cos \theta_{\vec{B}} & (\mathcal{B} + \kappa_g) \end{pmatrix}, \quad (31)$$

where

$$\begin{aligned}\mathcal{B} &= \frac{1}{\Omega^2 + 1} \\ \mathcal{C} &= -\frac{\Omega}{\Omega^2 + 1}.\end{aligned}\quad (32)$$

The other definitions are apparent by a term-by-term comparison of expressions in Eq. 30 and the corresponding entries in Eq. (31). All the z 's are the corrections that are small in $v_F q$. At $v_F q = 0$, the eigenmode equation $\text{Det}(1 + u\hat{\Pi}/2N_F) = 0$ has a solution $\Omega^2 + 1 = s_0 \equiv 2u - u^2$ such that $\Omega^2 = -(1-u)^2$. Restoring the dimensional frequency and continuing analytically to real frequencies, we find that the frequency at the $q = 0$ is simply given by $\Omega = \Delta_Z$, in agreement with Kohn's theorem. To find corrections to this result at small but finite $v_F q$, we look for a solution of the form $\Omega^2 + 1 = s_0 + \kappa_0$. Expanding the eigenmode equation to linear order in all the κ 's, we obtain

$$1 - 2\mathcal{B}u + (\mathcal{B}^2 + \mathcal{C}^2)u^2 = u[(1 - \mathcal{B}u)(\kappa_a + \kappa_b + \kappa_g) - 2u\mathcal{C}\kappa_c]. \quad (33)$$

Solving for κ_0 , we obtain

$$\kappa_0 = \frac{2(1-u)^2}{u} \frac{(v_F q)^2}{2}. \quad (34)$$

Restoring the units and continuing analytically, we obtain the frequency of the mode at finite q as

$$\Omega = \Delta_Z - \frac{(1-u)^2}{2u} \frac{(v_F q)^2}{\Delta_Z}. \quad (35)$$

Relabeling $u \rightarrow -F_0^a$, we obtain the coefficient a_2 , as given by Eq. (11) of the MT.

Chiral spin mode in the high-field limit

The frequency at $q = 0$

In this section, we derive the frequency of the chiral spin wave at $q = 0$ and in the high-field limit, when $\Delta_R, \Delta_D \ll \Delta_Z$ or $r, d \ll 1$ for dimensionless quantities. [Unless u is very close 1, i.e., the system is close to a ferromagnetic transition, there is no real difference between conditions $\Delta_R, \Delta_D \ll \Delta_Z$ and $\Delta_R, \Delta_D \ll \Delta_Z^*$.] With $\Pi_{ab}(i\Omega) \equiv \Pi_{ab}(i\Omega, \vec{q} = 0)$, we obtain from Eq. (28)

$$\frac{\Pi_{11}(i\Omega)}{-2N_F} = \frac{\sin^2 \theta_{\bar{B}}}{\Omega^2 + 1} \left[1 - 3 \frac{r^2 + d^2}{\Omega^2 + 1} + 2 \frac{r^2 + d^2 - 2rd \sin 2\theta_{\bar{B}}}{(\Omega^2 + 1)^2} \right] + \frac{1}{2} \frac{r^2 + d^2}{\Omega^2 + 1} + \frac{2rd \sin 2\theta_{\bar{B}}}{(\Omega^2 + 1)^2}, \quad (36)$$

$$\frac{\Pi_{22}(i\Omega)}{-2N_F} = \frac{\cos^2 \theta_{\bar{B}}}{\Omega^2 + 1} \left[1 - 3 \frac{r^2 + d^2}{\Omega^2 + 1} + 2 \frac{r^2 + d^2 - 2rd \sin 2\theta_{\bar{B}}}{(\Omega^2 + 1)^2} \right] + \frac{1}{2} \frac{r^2 + d^2}{\Omega^2 + 1} + \frac{2rd \sin 2\theta_{\bar{B}}}{(\Omega^2 + 1)^2},$$

$$\frac{\Pi_{33}(i\Omega)}{-2N_F} = \frac{1}{\Omega^2 + 1} \left[1 - 3 \frac{r^2 + d^2}{\Omega^2 + 1} + 2 \frac{r^2 + d^2 - 2rd \sin 2\theta_{\bar{B}}}{(\Omega^2 + 1)^2} \right] + \frac{r^2 + d^2}{\Omega^2 + 1} + \frac{4rd \sin 2\theta_{\bar{B}}}{(\Omega^2 + 1)^2},$$

$$\frac{\Pi_{12}(i\Omega)}{-2N_F} = -\frac{\sin \theta_{\bar{B}} \cos \theta_{\bar{B}}}{\Omega^2 + 1} \left[1 - 3 \frac{r^2 + d^2}{\Omega^2 + 1} + 2 \frac{r^2 + d^2 - 2rd \sin 2\theta_{\bar{B}}}{(\Omega^2 + 1)^2} \right] + \frac{rd}{\Omega^2 + 1} - \frac{2rd}{(\Omega^2 + 1)^2},$$

$$\frac{\Pi_{13}(i\Omega)}{-2N_F} = -\frac{\Omega \sin \theta_{\bar{B}}}{\Omega^2 + 1} \left[1 - 2 \frac{r^2 + d^2}{\Omega^2 + 1} + 2 \frac{r^2 + d^2 - 2rd \sin 2\theta_{\bar{B}}}{(\Omega^2 + 1)^2} \right] - \frac{2rd\Omega \cos \theta_{\bar{B}}}{(\Omega^2 + 1)^2},$$

$$\frac{\Pi_{23}(i\Omega)}{-2N_F} = \frac{\Omega \cos \theta_{\bar{B}}}{\Omega^2 + 1} \left[1 - 2 \frac{r^2 + d^2}{\Omega^2 + 1} + 2 \frac{r^2 + d^2 - 2rd \sin 2\theta_{\bar{B}}}{(\Omega^2 + 1)^2} \right] + \frac{2rd\Omega \sin \theta_{\bar{B}}}{(\Omega^2 + 1)^2}. \quad (37)$$

In a matrix form,

$$\frac{\hat{\Pi}(i\Omega)}{-2N_F} = \begin{pmatrix} \kappa_a + (\mathcal{B} + \kappa_b) \sin^2 \theta_{\bar{B}} & \kappa_f - (\mathcal{B} + \kappa_b) \sin \theta_{\bar{B}} \cos \theta_{\bar{B}} & (\mathcal{C} + \kappa_c) \sin \theta_{\bar{B}} + \kappa_d \cos \theta_{\bar{B}} \\ \kappa_f - (\mathcal{B} + \kappa_b) \sin \theta_{\bar{B}} \cos \theta_{\bar{B}} & \kappa_a + (\mathcal{B} + \kappa_b) \cos^2 \theta_{\bar{B}} & -(\mathcal{C} + \kappa_c) \cos \theta_{\bar{B}} - \kappa_d \sin \theta_{\bar{B}} \\ -(\mathcal{C} + \kappa_c) \sin \theta_{\bar{B}} - \kappa_d \cos \theta_{\bar{B}} & (\mathcal{C} + \kappa_c) \cos \theta_{\bar{B}} + \kappa_d \sin \theta_{\bar{B}} & 2\kappa_a + (\mathcal{B} + \kappa_b) \end{pmatrix}, \quad (38)$$

where \mathcal{B} and \mathcal{C} are the same as in Eq. (32) and the κ 's, which are small in r and d , are again defined by a term-by-term comparison of Eq. 37 and the entries in Eq. (38). We seek a solution of the form $\Omega^2 + 1 = s_0 + \kappa_0$, where z is also small in r and d . To linear order in the z 's, The eignemode equation reads

$$1 - 2\mathcal{B}u + (\mathcal{B}^2 + \mathcal{C}^2)u^2 = u \left[(1 - \mathcal{B}u)(2\kappa_b + 3\kappa_a - \kappa_f \sin 2\theta_{\bar{B}}) - 2u\mathcal{C}(\kappa_c + \kappa_d \sin 2\theta_{\bar{B}}) \right], \quad (39)$$

Solving for κ_0 we get:

$$\kappa_0 = (r^2 + d^2) \left\{ \frac{(1-u)(2-3u)}{2u} \right\} - rd \sin 2\theta_{\bar{B}} \left\{ \frac{(1-u)(2-u)}{u} \right\}. \quad (40)$$

Restoring the units and continuing analytically to real frequencies, we obtain the coefficients a_0 and \tilde{a}_0 in Eq. (11) of the MT.

A linear-in- q term in the dispersion

Now, we are interested in all terms to linear order in r, d and $v_F q$. Accordingly, we need to express to expand the quantities in Eq. (29) to linear order in these variables:

$$\begin{aligned} 2s_1 &= \Delta_Z^* (-\cos \theta_{\bar{B}} + r \sin \theta + d \cos \theta), \\ 2s_2 &= \Delta_Z^* (-\sin \theta_{\bar{B}} - r \cos \theta - d \sin \theta), \\ 4s_1^2 &= (\Delta_Z^*)^2 (\cos^2 \theta_{\bar{B}} - 2r \sin \theta \cos \theta_{\bar{B}} - 2d \cos \theta \cos \theta_{\bar{B}}), \\ 4s_2^2 &= (\Delta_Z^*)^2 (\sin^2 \theta_{\bar{B}} + 2r \cos \theta \sin \theta_{\bar{B}} + 2d \sin \theta \sin \theta_{\bar{B}}), \\ 4|s|^2 &= (\Delta_Z^*)^2 (1 - 2r \sin(\theta - \theta_{\bar{B}}) - 2d \cos(\theta + \theta_{\bar{B}})), \\ 4s_1 s_2 &= (\Delta_Z^*)^2 (\sin \theta_{\bar{B}} \cos \theta_{\bar{B}} + r \cos(\theta + \theta_{\bar{B}}) + d \sin(\theta - \theta_{\bar{B}})), \end{aligned} \quad (41)$$

Furthermore,

$$\begin{aligned}
\vec{v}_F \cdot \vec{q} &= v_F q \cos(\theta - \theta_{\vec{q}}) \\
\int_{\xi_{\vec{k}}} \int_{\omega} (g_+ \tilde{g}_+ + g_- \tilde{g}_-) &= \frac{2v_F q}{i\Omega} \cos(\theta - \theta_{\vec{q}}), \\
\int_{\xi_{\vec{k}}} \int_{\omega} (g_+ \tilde{g}_- + g_- \tilde{g}_+) &= -\frac{2}{\Omega^2 + 1} \left[1 - \frac{2\tilde{r}_\theta \Omega^2}{\Omega^2 + 1} \right] - \frac{2i\Omega v_F q}{(\Omega^2 + 1)^2} \left[\Omega^2 - 1 + \frac{2\tilde{r}_\theta(3\Omega^2 - 1)}{\Omega^2 + 1} \right] \cos(\theta - \theta_{\vec{q}}), \\
\int_{\xi_{\vec{k}}} \int_{\omega} (g_+ \tilde{g}_- - g_- \tilde{g}_+) &= \frac{2i\Omega}{\Omega^2 + 1} \left[1 - \frac{\tilde{r}_\theta(\Omega^2 - 1)}{\Omega^2 + 1} \right] + \frac{4\Omega^2 v_F q}{(\Omega^2 + 1)^2} \left[1 - \frac{\tilde{r}_\theta(\Omega^2 - 3)}{\Omega^2 + 1} \right] \cos(\theta - \theta_{\vec{q}}),
\end{aligned} \tag{42}$$

where $\tilde{r}_\vartheta \equiv r \sin(\vartheta - \theta_{\vec{B}}) + d \cos(\vartheta + \theta_{\vec{B}})$. Let's further introduce

$$\begin{aligned}
r_{1\vartheta} &\equiv r \sin \vartheta + d \cos \vartheta, \\
r_{2\vartheta} &\equiv r \cos \vartheta + d \sin \vartheta, \\
r_{3\vartheta} &\equiv r_{1\vartheta} \sin \theta_{\vec{B}} + r_{2\vartheta} \cos \theta_{\vec{B}}
\end{aligned} \tag{43}$$

such that $r_{1\vartheta} \cos \theta_{\vec{B}} - r_{2\vartheta} \sin \theta_{\vec{B}} = \tilde{r}_\vartheta$. Note that $(\tilde{r}_\vartheta, r_{3\vartheta})$ and $(r_{1\vartheta}, r_{2\vartheta})$ are related by a $\theta_{\vec{B}}$ rotation. This leads to:

$$\frac{\Pi_{11}(Q)}{-2N_F} = \frac{\sin^2 \theta_{\vec{B}}}{\Omega^2 + 1} \left[1 + \frac{i\Omega(3\Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_{\theta_{\vec{q}}} \right] + \frac{v_F q}{i\Omega} \frac{\Omega^2(3\Omega^2 + 1)}{2(\Omega^2 + 1)^2} r_{3\theta_{\vec{q}}} \sin 2\theta_{\vec{B}}, \tag{44}$$

$$\frac{\Pi_{22}(Q)}{-2N_F} = \frac{\cos^2 \theta_{\vec{B}}}{\Omega^2 + 1} \left[1 + \frac{i\Omega(3\Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_{\theta_{\vec{q}}} \right] - \frac{v_F q}{i\Omega} \frac{\Omega^2(3\Omega^2 + 1)}{2(\Omega^2 + 1)^2} r_{3\theta_{\vec{q}}} \sin 2\theta_{\vec{B}}, \tag{45}$$

$$\frac{\Pi_{33}(Q)}{-2N_F} = \frac{1}{\Omega^2 + 1} \left[1 + \frac{i\Omega(3\Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_{\theta_{\vec{q}}} \right], \tag{46}$$

$$\frac{\Pi_{12}(Q)}{-2N_F} = -\frac{\sin \theta_{\vec{B}} \cos \theta_{\vec{B}}}{\Omega^2 + 1} \left[1 + \frac{i\Omega(3\Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_{\theta_{\vec{q}}} \right] - \frac{v_F q}{i\Omega} \frac{\Omega^2(3\Omega^2 + 1)}{2(\Omega^2 + 1)^2} r_{3\theta_{\vec{q}}} \cos 2\theta_{\vec{B}}, \tag{47}$$

$$\frac{\Pi_{13}(Q)}{-2N_F} = -\frac{\Omega \sin \theta_{\vec{B}}}{\Omega^2 + 1} \left[1 + \frac{2i\Omega(\Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_{\theta_{\vec{q}}} \right] - \frac{v_F q}{i\Omega} \frac{\Omega^3 r_{3\theta_{\vec{q}}} \cos \theta_{\vec{B}}}{(\Omega^2 + 1)^2}, \tag{48}$$

$$\frac{\Pi_{23}(Q)}{-2N_F} = \frac{\Omega \cos \theta_{\vec{B}}}{\Omega^2 + 1} \left[1 + \frac{2i\Omega(\Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_{\theta_{\vec{q}}} \right] - \frac{v_F q}{i\Omega} \frac{\Omega^3 r_{3\theta_{\vec{q}}} \sin \theta_{\vec{B}}}{(\Omega^2 + 1)^2}. \tag{49}$$

In a matrix form,

$$\frac{\hat{\Pi}(Q)}{-2N_F} = \begin{pmatrix} \kappa_a \sin 2\theta_{\vec{B}} + (\mathcal{B} + \kappa_b) \sin^2 \theta_{\vec{B}} & -\kappa_a \cos 2\theta_{\vec{B}} - (\mathcal{B} + \kappa_b) \sin \theta_{\vec{B}} \cos \theta_{\vec{B}} & -(\mathcal{C} + \kappa_c) \sin \theta_{\vec{B}} - \kappa_d \cos \theta_{\vec{B}} \\ -\kappa_a \cos 2\theta_{\vec{B}} - (\mathcal{B} + \kappa_b) \sin \theta_{\vec{B}} \cos \theta_{\vec{B}} & -\kappa_a \sin 2\theta_{\vec{B}} + (\mathcal{B} + \kappa_b) \cos^2 \theta_{\vec{B}} & (\mathcal{C} + \kappa_c) \cos \theta_{\vec{B}} - \kappa_d \sin \theta_{\vec{B}} \\ (\mathcal{C} + \kappa_c) \sin \theta_{\vec{B}} + \kappa_d \cos \theta_{\vec{B}} & -(\mathcal{C} + \kappa_c) \cos \theta_{\vec{B}} + \kappa_d \sin \theta_{\vec{B}} & (\mathcal{B} + \kappa_b) \end{pmatrix}. \tag{50}$$

Once again \mathcal{B} and \mathcal{C} are the same as before and the κ 's are got by comparing Eqs. 44-49 and expression 50. We look for solution of the form $\Omega^2 + 1 = s_0 + \kappa_0$. To linear order in the κ 's, the eigenmode equation is now of the form

$$1 - 2\mathcal{B}u + (\mathcal{B}^2 + \mathcal{C}^2)u^2 = 2u \{ (1 - \mathcal{B}u)\kappa_b - u\mathcal{C}\kappa_c \} \tag{51}$$

Solving for κ_0 , we get

$$\begin{aligned}
\kappa_0 &= -\frac{2(1-u)^2[(4-u)(1-u) + u^2]}{u(2-u)^2} v_F q \tilde{r}_{\theta_{\vec{q}}} \\
&= -\frac{2(1-u)^2[(4-u)(1-u) + u^2]}{u(2-u)^2} v_F q (r \sin(\theta_{\vec{q}} - \theta_{\vec{B}}) + d \cos(\theta_{\vec{q}} + \theta_{\vec{B}})).
\end{aligned} \tag{52}$$

From here, one can read the coefficient a_1 as given by Eq. (11) of the MT.